The Marxian Transformation Problem – If it ain't broke, don't fix it

Presentation of the Marxian Transformation Problem

The Labour Theory of Value is one of the central elements of Marxist political economy, implying the exploitation of workers in a capitalist economy. It is not surprising that attempts have been made to demonstrate the supposed internal inconsistency of the theory. Several interpretations of one of the most difficult elements of the theory, the so-called "transformation problem", exist. These interpretations are summarised by Dumenil and Foley (2006) and by Kliman (2006). A formal definition of these interpretations in the language of linear algebra will be provided later in this document.

We would like to defend the Marxian Labour Theory of Value by demonstrating that the most widely accepted critique of the Marxian solution of the transformation problem, originating from Tugan-Baranovsky (1906) and Bortkiewicz (1907), is itself flawed. The alternative solution of the transformation problem, provided by Bortkiewicz (1907), was later incorporated by Neo-Ricardians into the Dual-System interpretation of the Marxian Labour Theory of Value (Pasinetti, 1977). According to Neo-Ricardians, the value of labour embedded in a commodity is not measured as the amount of socially necessary labour time required to produce the commodity but rather as the fraction of the value of the bundle of commodities consumed by the workers in order to sustain themselves (Pasinetti, 1977, p.123). We would argue that there is no need for the Dual-System interpretation if the original solution created by Marx is applied correctly. We would track the origins of the misinterpretation of Marx to virtually unknown among Western scholars work of Tugan-Baranovsky (1906), which is not available in English.

Marx claims that the only source of exchange value of commodities is labour. Workers are paid for labour-power, but their wages are not sufficient to buy all the commodities they make. A certain fraction of their working time is spent on producing commodities which will not be consumed by members of working class. The value of commodities produced during unpaid labour time has been called the surplus value. The rate of surplus value (the rate of exploitation), defined as the ratio of unpaid to paid labour time, is equalised between different sectors and firms, because workers can freely move between the industries seeking better working conditions. According to Marx, the exploitation of workers occurs during the production process but individual capitalists may not receive the exact share of the surplus value produced in their firms as the profit rate is equalised between different sectors, which can differ in the organic composition of capital (defined as the share of constant capital in the total capital outlay). Capital can flow between different industries as capitalists maximise their profits, which is assumed to lead to the equalisation of the profit rate. The process of realisation of surplus value leads to uniform allocation of profits through the system of prices of production, existing in the economy. Marx (1894, p.204) states that "*the sum of the profits in all spheres of production must equal the sum of the surplus-values, and the sum of the prices of production of the total social product equal the sum of its values*". It has to be assumed that values, profits and prices are measured in the same units, either money or labour-time.

We will try to understand what Marx meant by assuming that he knew what he was talking about, not by trying to discover supposed internal inconsistencies in his handwritten notes by taking various statements out of their context.

A simplistic reading of Marx might lead to the impression that he contradicted himself writing in some chapters of "Capital" that commodities always trade at their values and then writing in other places that they sell below or above their labour values, so that the rate of profit is equalised. Marx is aware of this problem and the contradiction is explained in the theory of transformation of surplus value into profits. According to its critics, the theory of transformation itself is internally inconsistent. Bortkiewicz (1907, p.201) claims that:

> *“This solution of the problem cannot be accepted because it excludes the constant and variable capitals from the transformation process, whereas the principle of the equal profit rate, when it takes the place of the law of value in Marx's sense, must involve these elements.”*

We intend to follow Moseley (2001), Dumenil and Foley (2006) in arguing that Marx did not make a mistake. We will argue that an error in the model of simple reproduction was introduced by Tugan-Baranovsky (1906). The numerical values from Tugan-Baranovsky’s model have been used in a more mature algebraic model presented by Bortkiewicz (1907). While Tugan-Baranovsky’s work is not well known to Western scholars as it is only available in Russian (and a German translation), the paper written by Bortkiewicz was later translated into English and published by Sweezy (1949).

If the internal consistency and validity of the original Marxian Labour Theory of Value is convincingly defended, this may amplify its impact on social science. The idea that a capitalist economy is based on the exploitation of workers, contradicted by the commonly accepted marginalist theory of value, would need to be seriously considered again. The Marxian Labour Theory of Value may help us understand the economic consequences of globalisation such as the deindustrialisation of some rich Western countries, as the minimisation of the cost of labour is often one of the factors determining where commodities are produced. It can answer the question of whether automation and artificial intelligence will steal jobs from workers. It may also provide the correct metrics for the planning of the transition to environmentally sustainable technologies. The Labour Theory of Value may provide a bridge linking Marxist, Neo-Ricardian and Post-Keynesian schools of economic thought.

In the paper we will provide a Post-Keynesian interpretation of the Marxian Labour Theory of Value, highlight the error made by Tugan-Baranovsky and Bortkiewicz in their critique of Marx and explain why the Neo-Ricardian dual-system interpretation of Labour Theory of Value is not consistent with the original ideas expressed by Marx and Engels. We would like to present the correct way of presenting the Marxian solution to the value transformation problem in a linear production model.

A few comments on Marxian Labour Theory of Value

Let us imagine a simple commodity production system existing in an economy. The only material inflows are natural resources and human labour. The material outflows are the final products and waste returned to the environment. The process of production of commodities requires the use of productive capital which is a stock. Productive capital consists of natural and produced components. Regardless of the system of social production all the output is a product of labour. It is assumed in the model that prices of commodities are “cost-determined” (Kalecki, 1954, p.11). We are not attempting to determine the values of objects which are not commodities, such as land or artworks, or address short-term fluctuations of prices of the commodities, which have "demand-determined" prices. The value of agricultural land can be linked with the value of capital but there exist categories of assets whose value is not covered by the Labour Theory of Value.

Marx (1867) claims that the exchange values of commodities depend on the amount of direct and indirect socially necessary labour required to make a unit of a commodity. It is assumed that complex labour can be reduced to simple labour and that all workers in the model have the same productivity. Marx (1867) also claims that in capitalism, workers are not paid the full value of their labour, however a fraction of the value, called a "surplus-value", is captured by capitalists in the process of self-expansion of the capital. Marx (1894, p.204-205) also explains that "*all capitals have the tendency, regardless of the surplus-value produced by them, to realise in the prices of their commodities the average profit, instead of their own surplus-value, in other words, to realise the prices of production*". We will try to determine the relationship between the values of commodities and prices in a capitalist economy. This task requires providing a clear definition of labour value.

In order to expose the relationship between labour values, surplus values, wages, prices and profits we need to make several assumptions simplifying the models. The government does not purchase goods or services but may redistribute profits. The economy is in a steady state, it is self-reproducing and the prices and wages are constant (the rate of inflation is equal to zero). The changes in technology and the creation of a surplus product, required for economic growth are therefore ignored. We will consider three simple models of a market economy:

1. A socialist system without capital markets and with zero profit rate,
2. A "classical" capitalist system with capital markets and a greater than zero profit rate,
3. A socialist system with capital markets and a greater than zero profit rate. Profits are fully redistributed to the workers by the state, acting as the agent of the working class. Capital markets are involved in allocation of production factors.

In both socialist market systems 1 and 3, workers are paid the full value of the labour-power they supply. We are ignoring changes in technology in "historic time" and the need to generate a surplus product if the economy is to grow. All the parameters describing the model except for the value of the stock of capital are flow variables. It is assumed that in all the models, workers supply the same amount of labour measured in time units "Λ_{lt}" and the same technology is used by firms in production of commodities.

The model of a socialist system without capital markets

The model is presented in Figure 1. The economy can be described as a socialist system without capital markets, with firms selling goods at their costs, not making any profits. Prices of commodities consist only of the cost of direct and indirect labour. Unit prices are equal to labour values of the commodities expressed in monetary units.

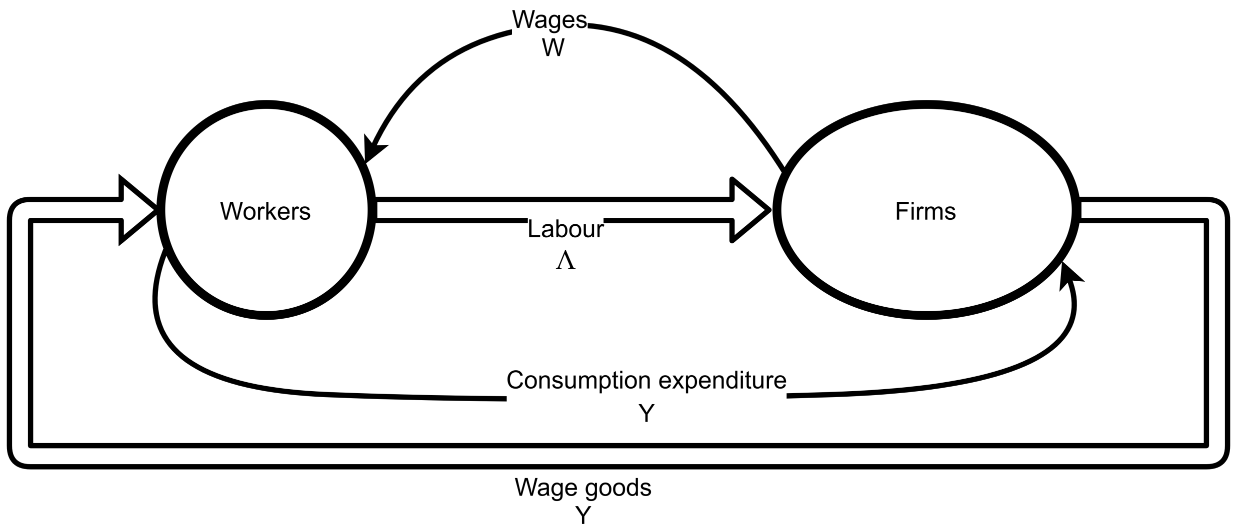

Figure 1: Real and monetary flows in a socialist economy without capital markets

Workers supply labour, which is measured in time units as "Λ_{lt}". Firms pay wages "W" (expressed in monetary units). Workers spend all their income on consumption, purchasing final goods "Y". The consumption spending is equal to the income. Since the firms do not generate profits, their aggregate costs, which only include wages W, are equal to the revenue from sales, which are equal to workers' consumption expenditure Y. Table 1 shows the economy's transaction flow matrix using the convention introduced by Godley and Lavoie (2007).

Table 1: Transaction flow matrix in a socialist economy without capital markets

	Workers	**Firms**
Consumption expenditures	-Y	Y
Wages	W	-W

In the model of a socialist system without capital markets the magnitude of flow of labour expressed in monetary units "Λ" is equal to wages "W" as workers are paid in full for the labour they supply. The "MELT" "μ" (Monetary Expression of Labour Time) is therefore equal to wage rate "w".

$$w\,\Lambda_{lt} = W = \Lambda = \mu\,\Lambda_{lt} \tag{1}$$

The following accounting identity holds in this economy:

$$\Lambda = W = Y \tag{2}$$

In a socialist system without capital markets, the sum of labour values expressed in monetary units is equal to the sum of the prices of production of the total net output (the value of final product, so-called Gross Domestic Product). The sum of profits and the sum of surplus values are both equal to zero.

The model of a capitalist system

The model is presented in Figure 2. In this system firms belong to capitalists who sell their products with profits. In the model of a self-reproducing economy, workers spend all their wages on wage goods while capitalists spend all their profits on luxury goods. The prices of commodities consist of costs of direct and indirect labour but also include profits distributed to capitalists. The mix of final products demanded by workers and capitalists in a capitalist system may differ from what is being purchased by workers in a socialist system without capital markets.

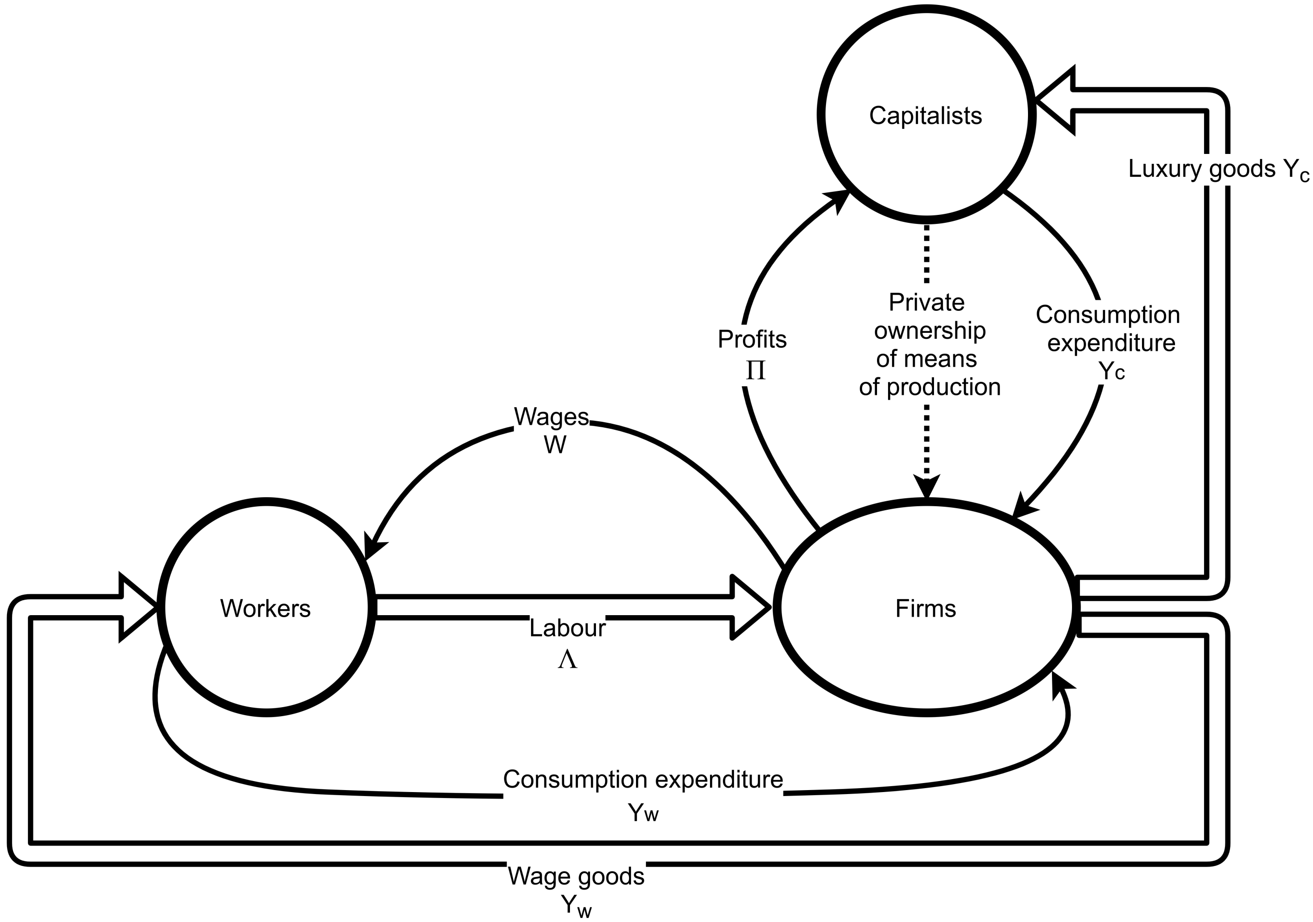

Figure 2: Real and monetary flows in a capitalist economy

The transaction matrix is shown in Table 2.

Table 2: Transaction flow matrix in a capitalist economy

	Workers	**Capitalists**	**Firms**
Consumption expenditure	$-Y_w$	$-Y_c$	$Y_w + Y_c$
Wages	W		-W
Profits		Π	$-\Pi$

The total value of the net output is equal to the sum of wages and profits. This equation is a social accounting identity.

$$W + \Pi = Y_w + Y_c = Y \tag{3}$$

This identity, applied to a more general case of growing economy, has been used by Kalecki (1954, p.45) as a starting point in the development of his Post-Keynesian theory of profits. (In fact, Kalecki's independent work on the theory of profits predates Keynesian General Theory but he is nevertheless considered a Post-Keynesian economist).

Wages are determined by the labour time and the wage rate. The sum of wages is lower than the total value of the net output as the sum of profits "Π" is greater than zero.

$$W = w\,\Lambda_{lt} \tag{4}$$

Marx (1867) claims that the value of labour "Λ" in a capitalist system (expressed in monetary units) is equal to the total value of the net output "Y" as no value is created "ex nihilo" in the process of production and exchange of commodities. According to Marx, the difference between the total value of labour and the wages (variable capital) is the surplus value, appropriated by capitalists in the process of circulation of capital. This is how much the workers are underpaid by capitalists. If surplus value is expressed in time units, it corresponds to unpaid time workers work for capitalists in the same way serfs used to work for landlords.

$$W + \Pi = W + S = \Lambda = Y \tag{5}$$

The opportunity to make profits arises in capitalism because the means of production are privately owned. Capitalists exercise their private property rights to the capital of firms; the commodities which are produced by the hired labour force belong to the owners of the enterprises, not to the workers who have physically produced them. Private property rights are defined and enforced by the state.

The statement that the sum of surplus values is equal to the sum of profits is also a social accounting identity.

$$\Pi = S \tag{6}$$

It is not something which needs to be, or even can be "proven". What can be debated is whether the categories of "total labour value Λ" and "surplus value S" are meaningful and whether they can be used to describe the economic reality of capitalism. Neoclassical economists claim that the wage rate "w" is equal to the marginal productivity of labour while the rate of profit is equal to the marginal productivity of capital. A "flow of value" from capital adds to the flow of labour. Shaikh (1974) demonstrates that the commonly used in neoclassical models Cobb-Douglas production function, which expresses the value of output as a function of monetary values of supplied capital and labour, is an artefact of the distribution of the revenue of firms between capitalists and workers. No value flows from capital when the rate of profit is equal to zero, as shown in Figure 1 and Table 1 even if the technology used in the production of commodities is the same as in a capitalist economy. Profits are determined by the social, not technical, conditions of production. We will further explore the relationship between the categories of labour value and monetary value of commodities in a model of a socialist system with capital markets.

The model of a socialist system with capital markets

The model is presented in Figure 3. A socialist system with capital markets can be established by nationalising privately-owned enterprises but leaving the capital markets intact, hoping that they will allocate the flow of investment more efficiently than the planning authority. The equity would belong to state-owned investment funds but shares of the companies would still be traded on the stock market as if they belonged to private investors. The profits would be returned to the state which would then redistribute them among workers as a social dividend. An alternative route to establish a socialist system with capital markets is implemented in China, where shares in state-owned enterprises are listed on stock exchanges.

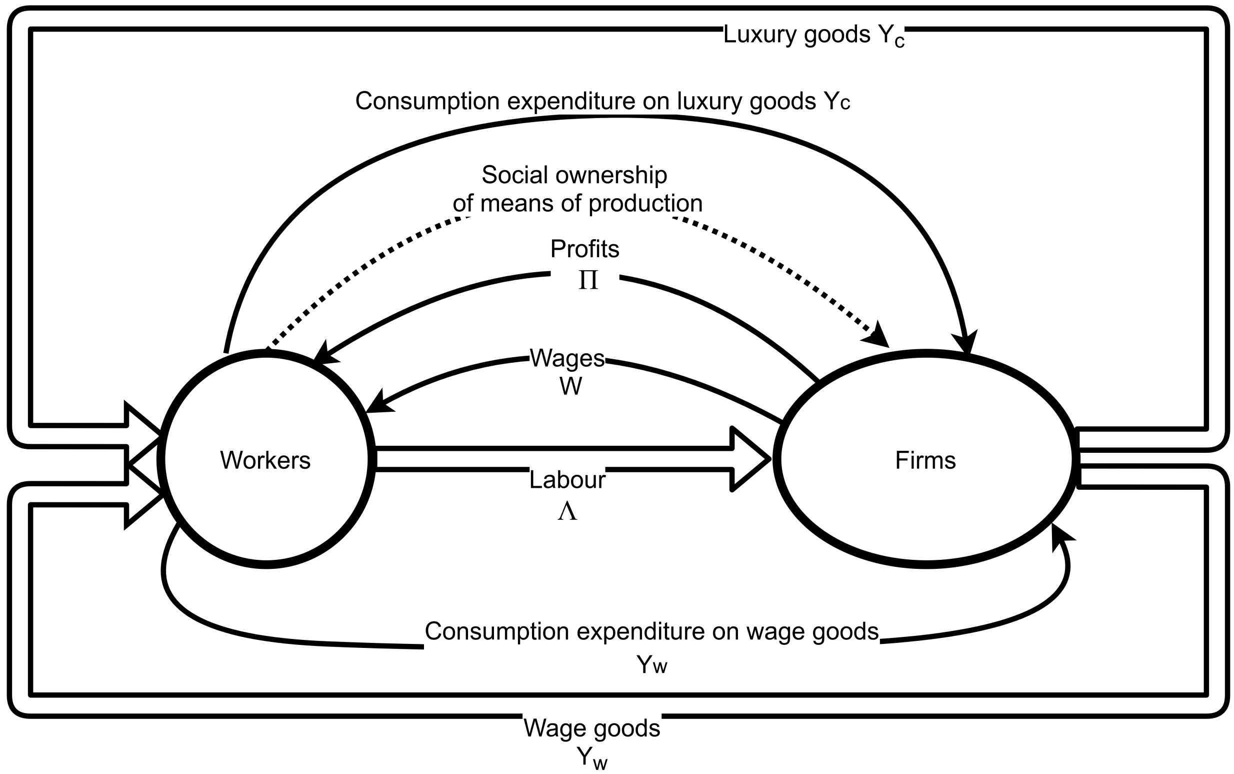

Figure 3: Real and monetary flows in a socialist economy with capital markets

Table 3: Transaction flow matrix in a socialist economy with capital markets

	Workers	**Firms**
Consumption expenditure	$-Y_w - Y_c$	$Y_w + Y_c$
Wages	W	-W
Profits	Π	-Π

The same social accounting identity (5), which applies to capitalism, describes the flow of exchange value in a socialist economy with capital markets.

We assume that the mix of the final product (in the form of wage and luxury goods) and the production technology are the same as in the capitalist economy. The wage rate and the rate of profit are also assumed to be the same. The prices of production of commodities are therefore identical to the previous example, as nothing has changed in the social sphere of production. The only difference between the systems is the elimination of capitalists as the class consuming luxury goods.

In a capitalist economy private owners of capital make profits proportional to the value of capital they own. It is assumed in the model that these profits are spent on acquiring luxury goods. In a socialist system (regardless of whether it has capital markets or not), different social-property relations allow workers to be paid the full value of the net product.

In a socialist system, workers are paid in full the monetary value of labour "Λ", even if the wage rate "w" differs from the "Monetary Expression of Labour time" (MELT). If a social dividend or a similar mechanism is implemented to redistribute the surplus value, transformed onto profits, back to workers, the sum of wages and redistributed profits is equal to the value of net product. It is evident that the full value of labour "Λ", expressed in monetary units is equal to the net product "Y", as this is how much workers are paid for their work and nobody else is involved in the production process. In capitalism workers are paid less and the difference is equal to the value of profits "Π". This, by definition, is equal to the surplus value "S".

The root of the Transformation Problem – the error in Tugan-Baranovsky's and Bortkiewicz's understanding of Marx's theory

The idea that Marx's Labour Theory of Value is internally inconsistent emerged almost immediately after the publication of Volume III of Capital (1894).

Bortkiewicz (1907), following Tugan-Baranovsky (1906), presents a model of 3-sector self-reproducing economy, producing intermediate, wage and luxury goods. Department (Sector) I produces capital (intermediate) goods using some preexisting intermediate goods and labour, Department II produces wage goods (consumed by workers) using some capital goods and labour, Department III produces luxury goods (consumed by capitalists) also using some intermediate goods and labour. The schema was defined by Tugan-Baranovsky (1906) in terms of monetary units (cost-prices, wages, profits and prices of production). It is assumed that all constant capital is circulating (there is no fixed capital in the system). We will use symbols from Dumenil and Foley (2006) as Tugan-Baranovsky (1906) used Cyrillic letters in his original paper and German symbols used by Bortkiewicz (1907) would be even less consistent with the vector notation introduced by Pasinetti (1977), which will be deployed in the last section of the paper. Equation (7) defines sums of prices of production in individual sectors:

$$\begin{aligned} c_1+v_1+\pi_1&=p_1\\ c_2+v_2+\pi_2&=p_2\\ c_3+v_3+\pi_3&=p_3 \end{aligned} \tag{7}$$

where "c_i" is the cost-price of constant capital, "v_i" is the variable capital (wages) expressed in monetary units, "π_i" is the profit and "p_i" is the sum of production prices of the commodities produced in the department "i" during a period of time.

Since the economy is self-reproducing, the system of equations (8) also must hold. Intermediate goods produced in period "t" are used in the production of wage goods and luxury goods in period "t+1". Wage goods and luxury goods produced in the period "t" are fully consumed by corresponding social classes in the period "t+1" This model describes an economy in a stationary state or, using neoclassical terminology, in equilibrium, so it is not necessary to specify time period indices in the equations. Kliman (2007) called this approach a method of "simultaneous valuation".

$$c_1+v_1+\pi_1=c_1+c_2+c_3=C$$
$$c_2+v_2+\pi_2=v_1+v_2+v_3=V \tag{8}$$
$$c_3+v_3+\pi_3=\pi_1+\pi_2+\pi_3=\Pi$$

Upper case symbols C, V and Π correspond to sums of constant capital, variable capital and profits across all sectors of the economy, expressed in monetary units.

Let us assume that labour values are also measured in monetary units, as in Marx (1885) Chapter XX. Labour values can be converted from labour-time to monetary units by multiplying them by the coefficient called the "Monetary Expression of Labour Time" (MELT), as in equation (48). The following equation describes labour values of the output of the economy's departments:

$$d_1+v_1+s_1=\lambda_1$$
$$d_2+v_2+s_2=\lambda_2 \tag{9}$$
$$d_3+v_3+s_3=\lambda_3$$

In the above formula, "d_i" is the labour value of the constant capital used in the department "i", "s_i" is the surplus value and "λ_i" is the labour value of the commodities produced during a period of time. The same values of variable capital (wages) "v_i" appear in both equations (8) and (9). It can be seen that monetary values of the variable capital (wages) are invariant to the transformation of surplus values onto profits. We have introduced separate symbols for labour values to avoid confusing them with cost-prices and prices of production.

Solving the transformation problem requires calculating values of "d_i" and "s_i" for i=1..3 from known "c_i", "v_i" and "π_i".

We will argue that Tugan-Baranovsky (1906) and Bortkiewicz (1907) have added an arbitrary assumption stating that the model of simple reproduction defined in terms of prices also has to self-reproduce itself in terms of labour values of the commodities produced by all the departments, even if the organic composition of capital in the departments differs from the average. Bortkiewicz (1907, p.200) claims that workers and capitalists consume the quantities of products whose labour values are equal to the value of variable capital and surplus value. This assumption is taken from a separate model of simple reproduction of social capital presented in the second volume of Capital (Marx, 1885). Bortkiewicz presents the following conditions of simple reproduction:

$$d_1+v_1+s_1=d_1+d_2+d_3$$
$$d_2+v_2+s_2=v_1+v_2+v_3 \tag{10}$$
$$d_3+v_3+s_3=s_1+s_2+s_3$$

The assumption made by Tugan-Baranovsky (1906) and Bortkiewicz (1907) leads to an algebraic inconsistency of the model if the organic composition of the capital is not the same in all the departments. Bortkiewicz (1907, p.201) claims that the Marx's "solution of the problem cannot be accepted because it excludes the constant and variable capitals from the transformation process, whereas the principle of the equal profit rate, when it takes the place of the law of value in Marx's sense, must involve these elements." The mistake is in merging two models created separately by Marx, the numerical example illustrating the transformation of surplus value into profits with a model of reproduction of social capital, despite the inconsistency of the assumptions.

Marx (1885, Chapter XX) presents a schema of simple reproduction of social capital, describing an economy fully reproducing itself over a period of time equal to the time of production. The schema of simple reproduction is defined in terms of labour values and organic composition of capital is identical in all departments. Marx explicitly mentions the possible divergence of labour values from prices of production.

> *It is furthermore assumed that products are exchanged at their value, and that no revolution in the value of the elements of productive capital takes place. Should there be any divergence of prices from values, this would not exert any influence on the movements of social capital. On the whole, there is the same exchange of the same quantity of products, although the individual capitalists would be taking shares in it which would no longer be proportional to their respective advances and to the quantities of value produced by each one.* (Marx, 1885, p.454-455)

The assumption that "products are exchanged at their values" no longer holds in the examples presented in Volume III, Chapter IX of Capital. Marx (1894, p.190) acknowledges that:

> *"so far as the variable capital is concerned, it is true that the average daily wage is equal to the values produced by the laborers in the time which they must work in order to produce their necessities of life. But this time is in its turn modified by the deviation of the prices of production of the necessities of life from their values."*

Marx was therefore aware that the labour values of wage goods may not be equal to the labour value of variable capital (wages) as prices of production can deviate from labour values.

Another error, made by Bortkiewicz (1907, p.201), is that he misinterpreted the concept of "sum of values" and "sum of prices of production" in a vertically integrated economy. Bortkiewicz wrote "*it emerges that the sum of these three price expressions, or the total price, is identical with the sum of the corresponding value expressions, or the total value*". But all the capital goods produced in the first department of the model economy are consumed in the process of reproduction of capital goods and the production of wage and luxury goods. If the value or the price of the output of the first department (that is, "intermediate goods") is added to the totals, double counting occurs. Marx (1894, p.189) warned against making this mistake: *"looking upon society as a whole, it would be a mistake to figure, say, the profit contained in the price of flax twice. It should not be counted as a portion of the price of linen and at the same time as the profit of the producers of flax.*"

The combination of these two errors made by Bortkiewicz leads to a model in which *"either aggregate labour is not the sole determinant of aggregate price, or aggregate unpaid labour is not the sole determinant of aggregate profit."*(Mohun, 1994, p.394)

In order to show the errors, we need to demonstrate the correct way of transforming the profits back to surplus values. Our goal is to calculate all the values of "d_i", "λ_i" and "s_i" appearing in equation (9) from the parameters specified in equation (7).

All the capital goods are made in the first department, their labour values must match:

$$d_1+d_2+d_3=\lambda_1 \tag{11}$$

The intermediate goods produced in the first department are homogeneous. There is no reason why their unit cost prices would not be the same in all the departments. The following conditions need to be met for labour values and prices of capital goods:

$$\frac{d_1}{c_1}=\frac{d_2}{c_2}=\frac{d_3}{c_3}=\alpha \tag{12}$$

where α is the ratio of labour value to price of constant capital.

The following equation holds for sum of the prices of production of the total social product "Y":

$$V+\Pi=Y \tag{13}$$

This equation can be also be interpreted as a social accounting identity stating that the gross domestic product in our model of the economy consists of the sum of wages (variable capital "V") and profits "Π".

We have to exclude the total price of constant capital "C" from the equation (13) as all the constant capital (intermediate goods) produced by the economy are consumed in the process of production of wage and luxury (final) goods. In the model, the sum of the prices of production of the total social product is equal to the sum of production prices of the wage-goods and luxury-goods, as these are final products.

$$Y=p_2+p_3 \tag{14}$$

Similarly, the following equation holds for total sum of labour values across all the departments:

$$V+S=\Lambda \tag{15}$$

where "S" is the total surplus value and "Λ" is the total labour value of the commodities produced in the economy during a period of time.

The labour value of constant capital (intermediate goods) disappears in this model as the quantity produced and used in the form of intermediate goods is incorporated into the quantity of final commodities produced.

$$\Lambda=\lambda_2+\lambda_3 \tag{16}$$

The average rate of profit "ρ" is defined as:

$$\rho=\frac{\Pi}{C+V} \tag{17}$$

Similarly, the rate of surplus value (rate of exploitation) "τ" is defined as:

$$\tau=\frac{S}{V} \tag{18}$$

Marx (1894) assumes that the rate of surplus value is identical in all of the departments as workers can move freely between firms:

$$\frac{s_1}{v_1}=\frac{s_2}{v_2}=\frac{s_3}{v_3}=\tau \tag{19}$$

It has also been assumed that the rate of profit is equalised between all departments because investors can move capital from firms offering a lower rate of profit to firms offering a higher rate of profit:

$$\frac{\pi_1}{c_1+v_1}=\frac{\pi_2}{c_2+v_2}=\frac{\pi_3}{c_3+v_3}=\rho \tag{20}$$

Finally, Marx (1894) claims that the labour value of the total social product in a period of time is equal to the sum of prices of production and the total surplus value is equal to the sum of profits in a period of time.

$$Y = \Lambda \tag{21}$$

$$S = \Pi \tag{22}$$

The equations (7) - (9) and (11) - (22) presented above contain all the information required to determine whether the numerical values used in the model are consistent and allow, in addition, for the calculation of surplus value.

We can determine the necessary conditions on values of "c_i" and "v_i" in a model of simple reproduction. This will also allow us to determine the profit rate as a function of monetary values of constant and variable capital involved in the production process. The information about the rate of profit and the rate of surplus value is embedded in the pricing system of the self-reproducing economy. Extracting this information will let us transform prices (monetary exchange values) onto labour values extracted in individual sectors (departments) of the economy.

Equation (7) can be rewritten to include the average rate of profit (17), as:

$$\begin{aligned} c_1+v_1+\rho(c_1+v_1) &= p_1 \\ c_2+v_2+\rho(c_2+v_2) &= p_2 \\ c_3+v_3+\rho(c_3+v_3) &= p_3 \end{aligned} \tag{23}$$

In a model of simple reproduction defined in terms of prices of production, (8) becomes:

$$\begin{aligned} c_1+v_1+\rho(c_1+v_1) &= c_1+c_2+c_3 \\ c_2+v_2+\rho(c_2+v_2) &= v_1+v_2+v_3 \\ c_3+v_3+\rho(c_3+v_3) &= \rho(c_1+c_2+c_3+v_1+v_2+v_3) \end{aligned} \tag{24}$$

which can be rearranged as:

$$\begin{aligned} c_1+v_1+\rho c_1+\rho v_1 &= c_1+c_2+c_3 \\ c_2+v_2+\rho c_2+\rho v_2 &= v_1+v_2+v_3 \\ c_3+v_3+\rho c_3+\rho v_3 &= \rho c_1+\rho c_2+\rho c_3+\rho v_1+\rho v_2+\rho v_3 \end{aligned} \tag{25}$$

and then:

$$\begin{aligned} v_1+\rho c_1+\rho v_1 &= c_2+c_3 \\ c_2+\rho c_2+\rho v_2 &= v_1+v_3 \\ c_3+v_3+\rho c_3+\rho v_3 &= \rho c_1+\rho c_2+\rho c_3+\rho v_1+\rho v_2+\rho v_3 \end{aligned} \tag{26}$$

The system of equations (26) is linearly dependent as the first equation is a sum of the second and the third.

$$c_2+\rho c_2+\rho v_2+c_3+v_3+\rho c_3+\rho v_3 = v_1+v_3+\rho c_1+\rho c_2+\rho c_3+\rho v_1+\rho v_2+\rho v_3 \tag{27}$$

The first equation in set (26) can be obtained by making the following simplifications to equation (27)

$$c_2+\cancel{\rho c_2}+\cancel{\rho v_2}+c_3+\cancel{v_3}+\cancel{\rho c_3}+\cancel{\rho v_3} = v_1+\cancel{v_3}+\rho c_1+\cancel{\rho c_2}+\cancel{\rho c_3}+\rho v_1+\cancel{\rho v_2}+\cancel{\rho v_3} \tag{28}$$

The linear dependency also allows us to drop one of the equations. We (arbitrarily) choose to drop the third equation and rearrange the remaining two:

$$\begin{aligned}\rho(c_1+v_1)&=c_2+c_3-v_1\\ \rho(c_2+v_2)&=v_1+v_3-c_2\end{aligned} \tag{29}$$

Since all the variables must be positive to have economic meaning, we can determine the profit rate from both equations. It has to be the same.

$$\rho=\frac{c_2+c_3-v_1}{c_1+v_1}=\frac{v_1+v_3-c_2}{c_2+v_2} \tag{30}$$

It is also possible to express the profit rate as a function of prices of production and variable or constant capital. The choice of prices of constant capital and wages (variable capital) as independent parameters is arbitrary.

We can now determine the rate of surplus value (exploitation) from known monetary values (prices) of constant and variable capital which appear in the model of simple reproduction. This is possible, knowing that "Π" (the sum of profits in the whole economy) has to be equal to "S" (the sum of surplus values expressed in monetary units in the whole economy).

From (17), (22) and (30) we can obtain:

$$S=\Pi=\frac{(c_2+c_3-v_1)(c_1+c_2+c_3+v_1+v_2+v_3)}{c_1+v_1} \tag{31}$$

The rate of surplus value (18) is then equal to:

$$\tau=\frac{(c_2+c_3-v_1)(c_1+c_2+c_3+v_1+v_2+v_3)}{(c_1+v_1)(v_1+v_2+v_3)} \tag{32}$$

Knowing the rate of exploitation, we can now determine the labour value of constant capital. From (9) we get:

$$\lambda_1-d_1=v_1+s_1 \tag{33}$$

Using (11) we can write that:

$$d_2+d_3=v_1+s_1 \tag{34}$$

Similarly, from (8) we get:

$$c_2+c_3=v_1+\pi_1 \tag{35}$$

The "rate of inverse transformation of constant capital" from (12) is then equal to:

$$\alpha=\frac{d_2+d_3}{c_2+c_3}=\frac{v_1+s_1}{v_1+\pi_1}=\frac{(1+\tau)v_1}{v_1+\pi_1} \tag{36}$$

We can then calculate values of constant capital from corresponding prices:

$$d_1=\alpha c_1$$
$$d_2=\alpha c_2 \quad (37)$$
$$d_3=\alpha c_3$$

Finally, (9) allows for calculating labour values of the output of the individual departments.

We can then compare our results of inverse transformation of profits into surplus values with the data produced by Tugan-Baranovsky (1906). The resulting labour values were subsequently used by Bortkiewicz (1907) in his numerical example illustrating the supposed internal inconsistency of Marxian theory.

The original numerical example of simple reproduction presented by Tugan-Baranovsky (1906, p.164) contained the following values:

$$180[c_1]+60[v_1]+60[\pi_1]=300[p_1]$$
$$80[c_2]+80[v_2]+40[\pi_2]=200[p_2] \quad (38)$$
$$40[c_3]+60[v_3]+25[\pi_3]=125[p_3]$$

We can see that the values used in the original model presented by Tugan-Baranovsky meet the condition (40) as the profit rate is the same in both linearly independent equations. The model is not internally inconsistent.

$$\rho=\frac{80+40-60}{180+60}=\frac{60+60-80}{80+80}=0.25 \quad (39)$$

Applying formula (32) to the original numerical example provided by Tugan-Baranovsky (1906) we obtain the following value of the rate of exploitation:

$$\tau=\frac{(80+40-60)(180+80+40+60+80+60)}{(180+60)(60+80+60)}=\frac{5}{8}=0.625 \quad (40)$$

This allows for calculating surplus values extracted in all departments:

$$s_1=0.625\cdot 60=37.5$$
$$s_2=0.625\cdot 80=50 \quad (41)$$
$$s_3=0.625\cdot 60=37.5$$

The numerical value of the coefficient α defined in (36) is equal to:

$$\alpha=\frac{1.625\cdot 60}{60+60}=\frac{13}{16}=0.8125 \quad (42)$$

This allows us to calculate labour values of constant capital:

$$d_1=0.8125\cdot 180=146.25$$
$$d_2=0.8125\cdot 80=65 \quad (43)$$
$$d_3=0.8125\cdot 40=32.5$$

Finally, labour values of products expressed in monetary units can be calculated using (9):

$$
\begin{array}{l}
146.25[d_1]+60[v_1]+37.5[s_1]=243.75[\lambda_1] \\
65[d_2]+80[v_2]+50[s_2]=195[\lambda_2] \\
32.5[d_3]+60[v_3]+37.5[s_3]=130[\lambda_3]
\end{array} \tag{44}
$$

We can validate these results.

$$Y=200+125=325=195+130=\Lambda \tag{45}$$

$$\Pi=60+40+25=125=37.5+50+37.5=S \tag{46}$$

In the transformed model, the sum of the prices of production of the total social product equals the sum of its values (45) and the sum of the profits in all departments equals the sum of the surplus-values (46).

Additionally, the labour values of constant capital used in all the departments are equal to the labour value of product of the capital goods sector, as specified in (11):

$$d_1+d_2+d_3=146.25+65+32.5=243.75=\lambda_1 \tag{47}$$

and (41) meets the condition specified in (19) (the rate of exploitation is identical in all the departments).

The model is now internally consistent in contrast to Tugan-Baranovsky's formulation (1906), as shown by equations (45) - (47).

In order to calculate the magnitudes of value in units of labour-time, Tugan-Baranovsky (1906) assumes that 150000 workers are employed in the first department and that the production period is equal to one year.

A coefficient called MELT (Monetary Expression of Labour Time) is required to convert labour values from labour-time to monetary units or in the opposite direction. This coefficient is a reciprocal of Mohun's (1994) "λ_m", the "value of money". In Tugan-Baranovsky's example, the MELT can be determined by dividing the labour value expressed in monetary units, added in the first department to the value of input commodities, by the amount of labour-time required to produce these commodities. In the calculation, surplus value must not be mistaken for the value of profits expressed in labour-time.

$$\mu=\frac{\lambda}{\lambda_{lt}}=\frac{(v_1+s_1)}{l_1} \tag{48}$$

In the above formula, "μ" is the MELT and "l" is the socially necessary labour time expended in the process of the production of commodities. The labour value "λ" of the produced commodities, measured in monetary units, is equal to "v+s".

In Tugan-Baranovsky's example, monetary units are millions of roubles. The value of MELT can be then determined as:

$$\mu=\frac{60000000+37500000}{150000}=650[roubles/year] \tag{49}$$

We can then present labour values corresponding to monetary values from the example, expressed in units of labour-time (as thousands of worker-years); values are rounded to 3 decimal places. The values in the system of equations (50) are obtained by dividing the corresponding values defined in (44) by the value of MELT "μ".

$$\begin{aligned}
&225[d_{lt1}]+92.308[v_{lt1}]+57.692[s_{lt1}]=375[\lambda_{lt1}]\\
&100[d_{lt2}]+123.077[v_{lt2}]+76.923[s_{lt2}]=300[\lambda_{lt2}]\\
&50[d_{lt3}]+92.308[v_{lt3}]+57.692[s_{lt3}]=200[\lambda_{lt3}]
\end{aligned} \tag{50}$$

The values of variable capital and surplus value presented by Tugan-Baranovsky (1906, p.164-166) are different:

$$\begin{aligned}
&225[d_{lt1}]+90[v_{lt1}]+60[s_{lt1}]=375[\lambda_{lt1}]\\
&100[d_{lt2}]+120[v_{lt2}]+80[s_{lt2}]=300[\lambda_{lt2}]\\
&50[d_{lt3}]+90[v_{lt3}]+60[s_{lt3}]=200[\lambda_{lt3}]
\end{aligned} \tag{51}$$

These values are used as an input in Bortkiewicz's (1907, p.204) algorithm transforming labour values into prices. Bortkiewicz considered them as correct values for a self-reproducing economy. The output of Bortkiewicz's procedure is identical to the values in (38) multiplied by 8/5 (to express prices in the same units as values).

The following reasoning has been presented by Tugan-Baranovsky (1906, p.165). (All the quotes from Tugan-Baranovsky's work have been translated by the author). In the first stage of the algorithm of transformation of prices into labour values, the labour value of capital goods produced in the first sector is determined by looking at the amount of labour required to produce these goods:

> *"Let us assume that in the first department there are employed 150 thousand workers. Using capital goods costing 180 million roubles these workers produce in one year commodities worth 300 million roubles. If we denote x as the labour value of these products, then the labour value of the capital goods, used in production of the products is equal to 180/300 x; what gives us the following equation:*
>
> $$\frac{180}{300}x+150000=x$$
>
> *from which we get that x=375000 (worker-years)".*

This value is correct as it does not depend on how much of the labour is paid (variable capital) and how much is unpaid (surplus value), it only depends on how much constant capital has been produced in the first department using the given amount of labour power. Since the output of the first sector is then divided between all three sectors as constant capital and the scaling factor is the same, the labour values of constant capital used in all the sectors are correct. The same scaling factor is applied to the commodities produced in other sectors of the economy, so their labour values are also correct.

The subsequent stages of the transformation algorithm have been described by Tugan-Baranovsky (1906, p.165) as:

> *"We will calculate the labour value of wage goods in the following way. The labour value of capital goods, used in the production of wage goods, is equal to* $375\cdot\frac{80}{300}$ *thousand worker-years, that is 100 thousand worker-years. The number of workers employed in this sector, which is related to the number of workers in the first sector as 80/60, is therefore equal to* $150\cdot\frac{80}{60}=200$ *thousand workers. The labour value of the products made in the second department is then equal to 100+200=300 thousand worker-years.*

In the third sector, the value of capital goods is equal to $375 \cdot \frac{40}{300} = 50$ *thousand worker-years. The number of workers who are employed in this sector is equal to the number of workers who are employed in the first sector and the labour value of the products is equal to 50+150=200 thousand worker-years."*

We can observe that labour values added in the process of production in all the sectors have been determined correctly. Tugan-Baranovsky (1906, p.165) makes however a mistake in calculating the rate of surplus value and as a consequence, in determining the surplus values extracted in all the departments:

The common rate of surplus value is equal to:

$$\frac{200\,(social\,surplus\,value)}{300\,(social\,variable\,capital)} = 66.6\,\%$$

This rate, according to the assumption, equally applies to all sectors of the economy."

After presenting numerical values contained in schema (51), Tugan-Baranovsky (1906, p.166) comments that "*these numbers express in thousands of worker-years the labour values of produced and consumed products*".

Later, Tugan-Baranovsky (1906, pp.166-167) concludes that:

"The rate of profit, calculated for prices, is equal to 25%, however expressed in terms of labour value, the total sum of surplus value reaches 200/675 which is almost 30% of the total value of capital. ... This way I have demonstrated that the general rate of profit does not coincide with the ratio of surplus value to the total value of capital."

The numbers, 200 and 300, used to calculate the rate of surplus value, correspond to the labour values of luxury goods and wage goods, consumed by capitalists and workers. Tugan-Baranovsky assumes that these values are equal to the surplus value extracted from the workers and the labour value of variable capital. As already mentioned, this assumption, taken from Volume II of Capital (Marx, 1885), may be incorrect. Workers and capitalists do not consume labour values but buy products spending their wages and profits at the prevailing prices which may diverge from labour values. This is the case in the example presented by Tugan-Baranovsky (1906).

Marx explains (1894, p.194) that if sectors of the economy have an organic composition of capital different from the average composition, prices of production will differ from labour values. If prices of production differ from labour values, the surplus value captured in the sector differs from the sum of profits realised in this sector. Wages and profits are not determined by the rate of exploitation but by the rate of profit.

The ratio of constant to variable capital is called a "value composition of capital". In the model created by Tugan-Baranovsky (1906), the value of organic composition of capital differs in every sector from the average value of the parameter:

$$\frac{C}{V} = \frac{180+80+40}{60+80+60} = \frac{3}{2}; \frac{c_1}{v_1} = \frac{180}{60} = 3; \frac{c_2}{v_2} = \frac{80}{80} = 1; \frac{c_3}{v_3} = \frac{40}{60} = \frac{2}{3} \tag{52}$$

The examples presented by Marx (1894) to illustrate the process of transformation of surplus value into profits in Volume III of Capital are not extensions of the simple reproduction model from Volume II (Marx, 1885). The schema of simple reproduction from Volume II, Chapter XX is defined in terms of labour values, and condition (10) is met, because the organic composition of capital is the same in all the departments.

While examining the divergence of prices of production from labour values of the commodities, it is incorrect to implicitly assume that prices have not diverged from the values and then point to the divergence of the sum of profits from the sum of surplus values.

We cannot therefore accept the validity of Tugan-Baranovsky's critique of the explanation proposed by Marx, of how the economy transforms surplus value to profits. We need to admit that Marx has not made a mistake in his procedure of transforming surplus value into profits. The model may or may not describe the real economy, but it is not internally inconsistent. Tugan-Baranovsky's critique is still influential today, as the main line of reasoning has been accepted by Bortkiewicz (1907), which later evolved into the Dual-System interpretation of Marxian Labour Theory of Value (Pasinetti, 1977).

The Transformation Problem in a linear production model

We will now move to Leontief linear production models, in order to investigate how to correctly apply the Marxian solution to the transformation problem in a model of physical commodity production and consumption (the 3-sector model introduced by Tugan-Baranovsky operates in a price-value space). We will build a monetary Marxian model, linking profits and surplus values in multiple industries.

The model allows for calculating prices of output commodities, knowing the relationship between quantities of input commodities and labour needed to produce unit quantities of output commodities within the given social production framework, described by the exogenous rate of profit (or rate of surplus-value). The quantities of final products have to be calculated outside of the production system.

Accepting the validity of the critique of Marx's Labour Theory of Value, Pasinetti (1977), following the earlier work of Sraffa, tries to determine unit prices and quantities of produced commodities in an equilibrium by combining together production and consumption in a single matrix including coefficients allowing for the reproduction of labour. In such a model, labour is effectively just another "intermediate commodity", like iron or electricity. This model is called a Leontief Closed System, as described by Pasinetti (1977). Neo-Ricardians express the value of commodities and labour in units of a "standard commodity". There is nothing wrong with the algebra used in these models but the fact that human labour is the only source of the exchange value is somehow obscured and has to be "rediscovered". If we want to build a monetary Marxian model, we would like to use the monetary value of a unit of labour, not a value of a bundle of commodities, as the unit of value. So-called labour values of commodities defined by Neo-Ricardians are not the same as labour values defined by Marx.

A modern economy is a monetary economy to which Say's law does not apply. We cannot ignore the functioning of the financial system and the role of the government sector. We therefore need to separate the production system from the rest of the economy generating consumption and investment demand. To build a linear model of the production system only, a Leontief Open System is required.

To fully describe an economy both the prices of all commodities and the quantities of produced commodities need to be determined. When constant returns to scale are assumed, the pricing system is independent to changes in produced quantities.

It is assumed in the model that only one type of labour is required in the production process or that complex labour can be reduced to simple labour. It is also assumed for simplicity that the production period is equal for all the commodities manufactured in the economy. Symbols consistent with Dumenil and Foley (2006) are used below, they differ from these used in the previous section.

The Marxian method of transforming surplus value into profits in an economy, presented in Marx (1894), is based on the following assumptions:

- the economy is a monetary economy, commodities are sold at prices of production which may differ from labour values
- the rate of exploitation is equal in the whole economy
- the rate of profit is equal in the whole economy
- the labour value of the total social product in a period of time is equal to the sum of prices of production of the total social product
- the sum of surplus values in the economy is equal to the sum of profits in a period of time

The economy produces "n" commodities during a period of production. A technique of production of a commodity "i" is characterised by a column vector **a**.

$$\boldsymbol{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \dots \\ a_{\mathrm{ni}} \end{bmatrix} \tag{53}$$

which contains the quantities of input commodities "1..n" required to produce a unit of commodity "i" and a scalar "l_i" describing the amount of labour required to produce a unit of commodity "i".

A technology is the collection of all techniques needed to produce all the commodities "1..n" which are produced by the economy and it is described by a matrix "**A**" (matrix of coefficients of production, obtained by combining together all the column vectors $\mathbf{a}_i$' for i=1..n) and the row vector "**l**" (vector of labour coefficients, consisting of scalar values "l_i" for i=1..n). The conditions which must be met by the matrix "**A**" in order to describe a self-reproducing system are presented by Morishima (1973, pp.21-27).

$$\boldsymbol{A} = \begin{bmatrix} a_{11}, a_{12}, .., a_{1n} \\ a_{21}, a_{22}, .., a_{2n} \\ .. \\ a_{n1}, a_{n2}, .., a_{nn} \end{bmatrix} \tag{54}$$

$$\boldsymbol{l} = [l_1, l_2, \dots, l_n] \tag{55}$$

Pasinetti (1977, p.60) calls these "a matrix of interindustry coefficients" and "a vector of direct labour coefficients".

A pattern of economic production (quantities of commodities j=1..n produced by the economy) is described by a column vector **x** (where x_j are also called levels of operation of techniques).

$$\boldsymbol{x}=\begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} \tag{56}$$

If the vector of output quantities "$\mathbf{x}_{out}$" is given, the vector of quantities "$\mathbf{x}_{inp}$" required in the process of production can be obtained from the following formula:

$$\boldsymbol{x}_{inp}=\boldsymbol{A}\,\boldsymbol{x}_{out} \tag{57}$$

The total labour-time "Λ_{lt}" required in the production process is equal to a scalar product of the vector of direct labour coefficients and the vector of levels of operation of techniques. We know that a scalar product is equal to a product of a row vector right-multiplied by a column vector, so these definitions are identical.

$$\Lambda_{\mathrm{lt}}=\boldsymbol{l}\cdot\boldsymbol{x}_{out}=\boldsymbol{l}\,\boldsymbol{x}_{out} \tag{58}$$

At this stage of building the model it is not determined which commodities are final commodities and which are non-final. The model hasn't been augmented to include fixed capital required in the production of commodities so its level of realism is limited. It is also assumed that turnover time is the same for all the production processes. A production process in a linear model of production is defined "backwards", that is, by back-projecting planned output quantities onto input quantities taken from the stocks of products made during previous cycles of production and determining the amount of labour (expressed as labour time) needed to complete the current production cycle. The demand for output commodities determines the demand for labour.

Dumenil and Foley (2006) introduce the concept of "classical" or "Ricardian" unit-values of commodities "λ_i" defined as the sum of the direct labour "l_i" expended in the production of a unit of commodity "i" and indirect labour used in the production of the intermediate commodities consumed during the production process, whose value is the sum of terms "$\lambda_i a_{ij}$" for all j=1..n. Pasinetti (1977, p.76) uses the term "vertically integrated labour coefficients" and identifies it with the Marxian "value" of a unit of a commodity. "Classical" unit values of commodities are usually measured in labour time per physical unit of a commodity.

$$\lambda_i=\lambda_1 a_{i1}+\lambda_2 a_{i2}+..+\lambda_n a_{\mathrm{in}}+l_i \tag{59}$$

The row vector of labour values of all commodities, defined as:

$$\boldsymbol{\lambda}=[\lambda_1,\lambda_2,..,\lambda_n] \tag{60}$$

satisfies the equation:

$$\boldsymbol{\lambda}=\boldsymbol{\lambda}\boldsymbol{A}+\boldsymbol{l} \tag{61}$$

The vector of labour values of units of commodities can be calculated by inverting the matrix (**I-A**) where **I** is the identity matrix:

$$\boldsymbol{\lambda}=\boldsymbol{l}(\boldsymbol{I}-\boldsymbol{A})^{-1} \tag{62}$$

We can then introduce the column vector of quantities of net product **y**. The elements corresponding to pure intermediate goods are equal to zero, while the elements corresponding to final goods have to be greater than zero to have correct economic meaning.

$$\boldsymbol{y}=\begin{bmatrix} y_1 \\ y_2 \\ .. \\ y_n \end{bmatrix} \tag{63}$$

The net product of the economy is equal to the total output from all the industries minus the commodities required for the replenishment of the intermediate commodities (57):

$$\boldsymbol{y}=\boldsymbol{x}-\boldsymbol{A}\boldsymbol{x} \tag{64}$$

Using the formula (64), Dumenil and Foley (2006) present the relationship between the vector of quantities of net product "**y**" and the vector of quantities of all commodities produced "**x**" in a self-reproducing or growing economy:

$$\boldsymbol{y}=(\boldsymbol{I}-\boldsymbol{A})\boldsymbol{x} \tag{65}$$

All the commodities not used for the production in the current cycle (as input or to replace the depreciated fixed capital) can be consumed or stored for the future use. In a growing economy investment becomes a part of the net product.

Dumenil and Foley (2006) assert that the labour value of the net product "Λ_{lt}" (that is the sum of labour values of all commodities) is equal to to the total labour time expended in the production of all the commodities.

$$\Lambda_{lt}=\boldsymbol{\lambda}\boldsymbol{y}=\boldsymbol{l}\boldsymbol{x} \tag{66}$$

To derive equation (66) we can left-multiply both sides of equation (65) by "**λ**" (knowing that a product of a row vector right-multiplied by a column vector is equal to the scalar product of these vectors).

$$\boldsymbol{\lambda}\boldsymbol{y}=\boldsymbol{\lambda}(\boldsymbol{I}-\boldsymbol{A})\boldsymbol{x} \tag{67}$$

We know that **λ** (the vector of labour unit-values) is determined in equation (62) by the technology, defined by **A** and **l**. This allows us to write:

$$\boldsymbol{\lambda}\boldsymbol{y}=\boldsymbol{l}(\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{I}-\boldsymbol{A})\boldsymbol{x} \tag{68}$$

Using the properties of an inverse matrix and then the identity matrix "**I**" we can drop the term "$(\mathbf{I}\text{-}\mathbf{A})^{-1}$ (**I**-**A**)", leading us to equation (66).

At this stage, Dumenil and Foley (2006) introduce the row vector of unit prices "**p**", the wage rate "w" and the rate of profit "ρ". The vector of unit prices is defined as:

$$\boldsymbol{p}=[p_1, p_2, .., p_n] \tag{69}$$

According to Marx (1894, p.186): *"The price of production of a commodity, then, is equal to its cost-price plus a percentage of profit apportioned according to the average rate of profit, or in other words, equal to its cost-price plus the average profit."* The unit cost-price of a commodity "i" is equal to the sum of the unit cost of direct labour and the unit cost of input commodities. The unit cost of direct labour is equal to the wage rate "w" multiplied by the direct labour coefficient "l_i". The unit cost of input commodities is determined by the technique of production "$\mathbf{a}_i$" and the vector of unit prices "**p**" (we are assuming a stationary economy where prices do not change). The unit price of commodity "i" is then equal to:

$$p_i = (1+\rho)(\boldsymbol{p}\,\boldsymbol{a}_i + w l_i) \qquad (70)$$

The vector of unit prices has to satisfy the following equation:

$$\boldsymbol{p} = (1+\rho)(\boldsymbol{p}\,\boldsymbol{A} + w\,\boldsymbol{l}) \qquad (71)$$

Pasinetti (1977, p.116) presents an alternative, Sraffian open Leontief system corresponding to the economy where workers are paid at the end of the production period and the monetary value of the variable capital is not included in profits. There is no fundamental difference between the behaviour of the Marxian and Sraffian systems. The vector of unit prices in a Sraffian system has to satisfy the following equation:

$$\boldsymbol{p} = (1+\rho)\,\boldsymbol{p}\,\boldsymbol{A} + w\,\boldsymbol{l} \qquad (72)$$

It is possible to extend the formula of cost pricing to include fixed capital, monopoly rents, value-added tax, etc. In a more general case, profit rates associated with specific commodities may differ and multiple types of labour (irreducible to simple labour) may be required to produce the commodities. Unit prices in a capitalist economy depend on technical and social conditions of production while labour values only depend on technology. Even in the most general case, unit prices do not depend on the final redistribution of the national income or whether the economy is growing or not. We do not need to know which commodities are consumed by workers as we are not trying to express the exchange values of commodities in relation to a bundle of commodities. The only assumption made in equation (71) about the self-reproducing economy is that the prices do not change during the cycle as the cost-prices in the current period of production are determined by the prices of production from the previous period. If this assumption was not met, we would need to construct an iterative (dynamic) process, describing the evolution of the production technology and the social conditions of production.

The vector of unit prices from equation (71) can be determined in the following way:

$$\boldsymbol{p} = (1+\rho)\,\boldsymbol{p}\,\boldsymbol{A} + (1+\rho)\,w\,\boldsymbol{l} \qquad (73)$$

$$\boldsymbol{p}[\boldsymbol{I} - (1+\rho)\boldsymbol{A}] = (1+\rho)\,w\,\boldsymbol{l} \qquad (74)$$

$$\boldsymbol{p} = (1+\rho)\,w\,\boldsymbol{l}[\boldsymbol{I} - (1+\rho)\boldsymbol{A}]^{-1} \qquad (75)$$

We will try to determine how surplus value is redistributed in the system of social production described by equation (71). Instead of defining the bundle of wage-commodities and expressing the wage bill in terms of the scalar product of the vector of quantities of wage-commodities and the vector of prices, we will determine the scale of production in terms of the net product of the economy "**y**" and use social accounting identities to establish the relationship between the rate of profit and the rate of surplus value.

The monetary value of net product "Y" (GDP) depends on the quantities of commodities in the net product bundle and unit prices of the commodities:

$$Y = \boldsymbol{p}\,\boldsymbol{y} \tag{76}$$

As stated in equation (13), net product is equal to the sum of wages "W" (variable capital "V") and profits "P". The wage bill "W" is determined by the wage rate "w" and the amount of socially necessary labour expended during the current production period "Λ_{lt}" (the labour value of net product expressed in time units).

$$W = w\,\Lambda_{lt} = w\,\boldsymbol{l}\,\boldsymbol{x} \tag{77}$$

In a Marxian model, the rate of surplus value (rate of exploitation) "τ" defined in equation (19) as the ratio of the total surplus value "S" to variable capital "V", has to be equal to the ratio of the sum of profits "P" to the wage bill "W" as stated in the second "Marxian equation" (22).

$$\tau = \frac{P}{W} \tag{78}$$

We can assume that in a market socialist system, all the value added during the production period would be distributed among workers (Y=W, P=0). We may also assume that the government would only apply a tax on wages to offset some or all of its expenditures, so that no disruption to the pricing system would occur. In this case, "direct prices" mentioned by Dumenil and Foley (2006, p.11) would manifest themselves in the economy, if all the commodities are sold at their production costs. From (76) and (77) we get:

$$\boldsymbol{p}\,\boldsymbol{y} = w\,\boldsymbol{l}\,\boldsymbol{x} \tag{79}$$

The vector of unit prices "**p**" in a market socialist economy can be calculated from (75) assuming that the profit rate ρ=0.

$$\boldsymbol{p} = w\,\boldsymbol{l}\,(\boldsymbol{I} - \boldsymbol{A})^{-1} \tag{80}$$

Unlike in a market socialist system, in a capitalist economy the rate of surplus value $\tau > 0$. Using (78), we can present the monetary value of the net product defined in (13) as a function of wages and the rate of surplus value:

$$Y = (1 + \tau)\,W \tag{81}$$

We can combine equations (76) (the GDP expressed as the volume of aggregate supply, the price of the bundle of final goods sold in a unit of time) and (81) (the GDP expressed as the aggregate demand, equal to the sum of wages and profits-dividends paid in a unit of time). It is assumed that in a self-reproducing economy these have to be equal as all the income is consumed. We will be able to discover the relationship between the rate of profit which appears in the pricing equation and the rate of surplus value which appears in the social income distribution equation.

$$\boldsymbol{p}\,\boldsymbol{y} = (1 + \tau)\,W \tag{82}$$

The vector of unit prices has been determined in equation (75) and the wage bill "W" has been determined in equation (77).

$$(1+\rho)w\boldsymbol{l}[\boldsymbol{I}-(1+\rho)\boldsymbol{A}]^{-1}\boldsymbol{y}=(1+\tau)w\boldsymbol{l}\,\boldsymbol{x} \tag{83}$$

From equation (65) we can calculate the value of the vector of quantities of produced commodities "**x**":

$$\boldsymbol{x}=(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{y} \tag{84}$$

We can then substitute "**x**" in equation (83).

$$(1+\rho)w\boldsymbol{l}[\boldsymbol{I}-(1+\rho)\boldsymbol{A}]^{-1}\boldsymbol{y}=(1+\tau)w\boldsymbol{l}(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{y} \tag{85}$$

Equation (85) can be simplified by dropping the wage rate "w".

$$(1+\rho)\boldsymbol{l}[\boldsymbol{I}-(1+\rho)\boldsymbol{A}]^{-1}\boldsymbol{y}=(1+\tau)\boldsymbol{l}(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{y} \tag{86}$$

Finally, the rate of surplus value in a self-reproducing economy, producing the quantities of net product "**y**" with technology described by "**A**" and "**l**" can be determined as a function of profit rate "ρ" as:

$$\tau=(1+\rho)\frac{\boldsymbol{l}[\boldsymbol{I}-(1+\rho)\boldsymbol{A}]^{-1}\boldsymbol{y}}{\boldsymbol{l}(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{y}}-1 \tag{87}$$

From equations (66) and (77) the sum of surplus values expressed in monetary units "S" is equal to:

$$S=\frac{\tau}{1+\tau}\mu\boldsymbol{\lambda}\,\boldsymbol{y} \tag{88}$$

The formula for the sum of profits "P" includes the sum of monetary values of variable capital and the sum of prices of constant capital (the total monetary value of the capital advanced), multiplied by the profit rate.

$$P=\rho(w\,\boldsymbol{l}\,\boldsymbol{x}+\boldsymbol{p}\,\boldsymbol{A}\,\boldsymbol{x}) \tag{89}$$

The first Marxian social accounting identity (the sum of profits is equal to the sum of surplus values), introduced in equation (6) and restated as equation (22), can be written as:

$$S=\frac{\tau}{1+\tau}\mu\boldsymbol{\lambda}\,\boldsymbol{y}=\rho(w\,\boldsymbol{l}\,\boldsymbol{x}+\boldsymbol{p}\,\boldsymbol{A}\,\boldsymbol{x})=P \tag{90}$$

The second Marxian social accounting identity (the sum of the prices of production of the net social product is equal to the sum of its values) has been introduced in equation (5) and restated as equation (21). The formula for the total monetary value of labour performed in a unit of time "Y", can be derived from equations (66) and (76).

$$\Lambda=\mu\boldsymbol{\lambda}\,\boldsymbol{y}=\mu\boldsymbol{l}\,\boldsymbol{x}=\boldsymbol{p}\,\boldsymbol{y}=Y \tag{91}$$

Labour value in the Dual-System interpretation of Marxian Labour Theory of Value

The dual-system interpretation is presented by Pasinetti (1977, pp.122-150). The author defines Marxian labour values of commodities in terms of total socially necessary labour time required to produce physical units, as in equation (62). These labour values manifest themselves as (monetary) unit prices in a socialist system without capital markets, depicted in Figure 1 and defined by equation (1).

This system of unit prices is described by the following equation, which is a special case of equation (72), with the rate of profit ρ equal to 0:

$$\boldsymbol{p}=\boldsymbol{p}\,\boldsymbol{A}+w\,\boldsymbol{l} \tag{92}$$

The solution of this equation has been already shown in equation (80) as it is also a special case of the general price equation (71) which is applicable to all self-reproducing market economic systems described by Leontief linear production models.

Equation (62) appears in Pasinetti (1977, p.122) as (V.A.1), the vector of direct labour coefficients "**l**" is denoted there as "$\mathbf{a}_n$" and the vector of labour values "$\boldsymbol{\lambda}$" as "**v**". Pasinetti considers unit prices expressed in monetary units to be "relative" and normalises them by setting the wage rate "w" to 1. Since in a socialist system without capital markets workers are paid the full value of their labour as wages, the wage rate "w" is equal to "Monetary Expression of Labour Time" (MELT) "μ" and denoted by Pasinetti as "w^*". If MELT is set to be equal to 1, unit prices expressed in monetary units are equal to socially necessary labour time expressed in time units.

If we want to compare labour values and prices, we need to measure labour values in monetary units. We will use the symbol "**v**" to denote the row vector or labour values.

$$\boldsymbol{v}=\mu\,\boldsymbol{\lambda} \tag{93}$$

The following equation is then the equivalent of equation (61)

$$\boldsymbol{v}=\boldsymbol{v}\,\boldsymbol{A}+\mu\,\boldsymbol{l} \tag{94}$$

The column vector "$\mathbf{l}_m$" corresponds to monetary values of direct labour required to produce the commodities described by the vector "**l**".

$$\boldsymbol{l}_{\boldsymbol{m}}=\mu\,\boldsymbol{l} \tag{95}$$

The following vector of labour values satisfies equation (94)

$$\boldsymbol{v}=\boldsymbol{l}_{\boldsymbol{m}}(\boldsymbol{I}-\boldsymbol{A})^{-1} \tag{96}$$

This equation appears in Pasinetti (1977, p.123) as (V.A.5), it is also identical to equation (80).

Pasinetti (1977, p.123) mentions that according to Marx, in a capitalist system the owners of the means of production are able to pay workers less than the full value of their labour. The fraction of the "complete" wage paid to workers in capitalism in the form of wages can be denoted by "δ". Pasinetti introduces a column vector of quantities of wage goods purchased by workers for their wages earned in a unit of time, denoted by "**d**". Not all the goods produced in the economy are wage goods, so some elements of the vector are equal to zero.

$$\boldsymbol{d}=\begin{bmatrix} d_1 \\ d_2 \\ .. \\ d_h \\ 0 \\ .. \\ 0 \end{bmatrix} \tag{97}$$

"**d**" is a subsistence real wage rate expressed in physical units of wage goods. The monetary value of wages paid for a unit of labour time (the wage rate) can be then expressed as a scalar product of the vector of labour values of commodities measured in monetary units "**v**" and the vector "**d**":

$$w_v = \boldsymbol{v}\,\boldsymbol{d} = \delta w^*; 0 \leqslant \delta < 1; w^* = 1 \qquad (98)$$

The wage rate defined in equation (98) "w_v" is equal to the fraction "δ" of the MELT "μ" (denoted as "w^*"), if we set MELT "μ" to 1, the monetary wage rate is equal to "δ". Equation (98) corresponds to equation (V.A.7) in Pasinetti (1977, p.124).

According to Pasinetti, the following equation describes the system of distribution of unit labour values emerging in capitalism:

$$\boldsymbol{v} = \boldsymbol{v}\,\boldsymbol{A} + \frac{1}{\delta}\boldsymbol{v}\,\boldsymbol{d}\,\boldsymbol{l_m} \qquad (99)$$

This equation is derived from equations (92) and (94) by substituting for the "complete" wage rate "w^*" the expression "w_v/ δ" which is then derived from equation (98).

It is assumed that:

$$\frac{1}{\delta}\boldsymbol{v}\,\boldsymbol{d} = 1 \qquad (100)$$

We can also derive equation (99) by multiplying the vector of direct labour coefficients "$\mathbf{l}_\mathrm{m}$" by the expression defined in formula (100).

The coefficient "δ" is expressed as a function of the rate of surplus value "τ" (Pasinetti uses the symbol "σ").

$$\tau = \frac{1}{\delta} - 1 \qquad (101)$$

Equation (99) can be then rearranged as:

$$\boldsymbol{v} = \boldsymbol{v}\,\boldsymbol{A} + (1+\tau)\boldsymbol{v}\,\boldsymbol{d}\,\boldsymbol{l_m} \qquad (102)$$

The following assumption introduced in equation (100) is still valid:

$$(1+\tau)\boldsymbol{v}\,\boldsymbol{d} = 1 \qquad (103)$$

The expression "$\mathbf{dl}_\mathrm{m}$" is a square matrix of the same dimension as "**A**", generated by the multiplication of two vectors.

Equation (102) can be rearranged to describe a system reproducing itself in terms of unit labour values over a period of production. The matrix "[**A**+(1+τ)$\mathbf{dl}_\mathrm{m}$]" defines the linear transformation of vector "**v**" into itself.

$$\boldsymbol{v} = \boldsymbol{v}[\boldsymbol{A} + (1+\tau)\boldsymbol{d}\,\boldsymbol{l_m}] \qquad (104)$$

Pasinetti (1977, p.125) rearranges equation (102) further, presenting the following system of system of linear homogeneous equations:

$$\boldsymbol{v}[(\boldsymbol{I} - \boldsymbol{A}) - (1+\tau)\boldsymbol{d}\,\boldsymbol{l_m}] = \mathbf{0} \qquad (105)$$

A necessary condition for the existence of nonzero solutions of equation (105) is:

$$det[(\boldsymbol{I}-\boldsymbol{A})-(1+\tau)\boldsymbol{d}\,\boldsymbol{l_m}]=0 \tag{106}$$

According to Pasinetti, if the vector "**d**" (a subsistence real wage rate expressed in physical units of wage goods) is given, it is possible to determine the rate of surplus value "τ" and then determine the vector of labour values "**λ**". We know that labour values of commodities are only determined by technical, not social, costs of production. Labour values do not depend on who consumes the economic output. Equation (61), rearranged by Pasinetti to include the real subsistence wage rate and the rate of surplus value as equation (105), has already been resolved by inverting the matrix "(**I-A**)". The solution is presented in equation (62), mentioned as (V.A.5). There is no need to express the value of labour in terms of the sum of labour values of the commodities produced and consumed by separate social classes which may lead to confusing surplus value and surplus product. The adequate measure of the labour value is the socially necessary labour time or its monetary equivalent.

Pasinetti (1977, p.125) presents the conditions of self-reproduction of a capitalist economy derived from equation (102) by introducing the vector of output quantities "**x**" ("the pattern of economic production"). This vector is denoted by Pasinetti in equation (V.A.15) by symbol "$\mathbf{Q}^{\sim}$".

$$\boldsymbol{v}\,\boldsymbol{x}=\boldsymbol{v}\,\boldsymbol{A}\,\boldsymbol{x}+\boldsymbol{v}\,\boldsymbol{d}\,\boldsymbol{l}_m\boldsymbol{x}+\tau\boldsymbol{v}\,\boldsymbol{d}\,\boldsymbol{l}_m\boldsymbol{x} \tag{107}$$

The macroeconomic interpretation given by Pasinetti to equation (107) is that the sum of labour values of the total gross product "**λx**" is equal to the sum of values of commodities required to replace the means of production "**λAx**", the sum of values of commodities required to "replace" the labour force (the value of the subsistence wages) "**λdlx**" and the sum of surplus values "**τλdlx**".

As already mentioned, this definition of "surplus value" used by Pasinetti differs from the definition introduced by Marx (1867, p.168) in the context of the process of circulation of capital M-C-M' "*where M'=M+ΔM*". The "*increment*" or "*excess over the original value*" clearly applies to monetary capital, present at the start and the end of the capital circulation cycle (the deviation of individual surplus values from individual profits is precisely the subject of the transformation debate). Marx (1867, p.241) also states that surplus value is "*nothing but materialised surplus-labour*" expended during "*surplus labour-time*". It is therefore the amount of labour expended during unpaid labour-time, not the amount of labour embedded in the commodities bought by capitalists after realising their profits. Pasinetti (1977) accepts the erroneous view expressed by Bortkiewicz (1907) that workers and capitalists consume the quantities of products whose labour values are equal to the value of variable capital (wages) and surplus value.

At this stage, the "price-of-production system" described by equation (73), referred to by Pasinetti (1977, p.126) as (V.A.16), is introduced (Pasinetti used the symbol "π" instead of "ρ" to denote the rate of profit). The wage rate "w" is substituted by the following equation, corresponding to equation (98). The vector "**d**" is obviously the same as in the "value" system as they are describing the same system. Let us denote the wage rate defined in the "price-of-production system" by "w_p".

$$w_p=\boldsymbol{p}\,\boldsymbol{d} \tag{108}$$

This substitution leads to the following equation describing the "price-of-production system" in terms of real wage rate expressed in physical units of wage goods:

$$\boldsymbol{p}=(\boldsymbol{p}\,\boldsymbol{A}+\boldsymbol{p}\,\boldsymbol{d}\,\boldsymbol{l})(1+\rho) \tag{109}$$

The equation (109) implicitly uses the same MELT "μ" equal to one so that the vector "$\mathbf{l}_m$" describing the monetary values of direct labour required to produce unit quantities of commodities is numerically equal to the vector "**l**", describing the direct labour coefficients expressed in labour-time per unit of commodity. These vectors are expressed in different units, but this does not cause any problems as long as the assumption regarding the value of MELT made for the "value-system" is also applied to the "price-system".

We can rearrange equation (109) so that it describes a system of unit-prices reproducing itself over a period of production. The matrix "(**A**+**dl**)(1+ρ)" defines the linear transformation of vector "**p**" into itself.

$$\boldsymbol{p}=\boldsymbol{p}(\boldsymbol{A}+\boldsymbol{d}\,\boldsymbol{l})(1+\rho) \qquad (110)$$

Pasinetti (1977, p.126) rearranges equation (109) further into a system of linear homogeneous equations (V.A.19).

$$\boldsymbol{p}[\boldsymbol{I}-(1+\rho)(\boldsymbol{A}+\boldsymbol{d}\,\boldsymbol{l})]=\boldsymbol{0} \qquad (111)$$

The matrix of interindustry coefficients "**A**" has been modified to include the consumption of commodities (wage goods) required to replenish the labour force by adding the term "**dl**".

$$\boldsymbol{A}^{(+)}=\boldsymbol{A}+\boldsymbol{d}\,\boldsymbol{l} \qquad (112)$$

The characteristic equation of augmented matrix "$\mathbf{A}^{(+)}$" (111) is:

$$det[\boldsymbol{A}^{(+)}-\frac{1}{(1+\rho)}\boldsymbol{I}]=0 \qquad (113)$$

According to Pasinetti (1977, pp.127-128), equation (113) presents a necessary condition of the existence of non-zero solutions of equation (111). The rate of profit can be determined from the maximum eigenvalue of matrix "$\mathbf{A}^{(+)}$", which is assumed to be less than or equal to one in order to ensure that the rate of profit is equal to or greater than zero. The calculation of the rate of profit allows for the determination of the vector of unit prices "**p**", which is the dominant eigenvector, corresponding to the dominant eigenvalue determined by solving the characteristic equation defined by equation (113).

We have already determined "**p**" without expressing the wage rate as a vector of quantities of wage-goods bought for a unit or labour time, by inverting the matrix "[**I**-(1+ρ)]**A**]" in equation (75). The vector of unit prices "**p**" depends on the rate of profit "ρ" and the wage rate "w".

Pasinetti (1977, p. 129) rearranges characteristic equations (106) and (113) so that they can be more easily compared.

$$det[\frac{1}{(1+\tau)}\boldsymbol{I}-(\frac{1}{(1+\tau)}\boldsymbol{A}+\boldsymbol{d}\,\boldsymbol{l})]=0 \qquad (114)$$

$$det[\frac{1}{(1+\rho)}\boldsymbol{I}-(\boldsymbol{A}+\boldsymbol{d}\,\boldsymbol{l})]=0 \qquad (115)$$

If both "τ" and "ρ" are greater than zero, dominant eigenvalues and dominant eigenvectors of value and price systems could be different. Pasinetti (1977, p.129) observes that in general (except for special cases such as uniform “organic composition of capital”), “**v**” is not equal to “**p**”. From this he infers that in general,

$$\boldsymbol{v}\,\boldsymbol{x} \neq \boldsymbol{p}\,\boldsymbol{x} \tag{116}$$

The statement that the valuation of total product in terms of labour-values may differ from the valuation in terms of prices of production does not contradict the statement that “*the sum of the prices of production of the total social product equal the sum of its values”* (Marx 1894, p.204) as the “total social product” is the net product “**y**” (also called the Gross Domestic Product) not the vector of output quantities “**x**”. Some of the commodities included in “**x**” are intermediate goods and their value is excluded from the calculation of the GDP to avoid double counting.

The total monetary value of labour performed in a unit of time “Y” has already been presented in equation (91). It has been demonstrated that the following assumptions, based on the social accounting model, do not lead to any contradictions between the Marxian labour value and price systems.

$$Y = \mu\,\boldsymbol{\lambda}\,\boldsymbol{y} = \boldsymbol{v}\,\boldsymbol{y} = \mu\,\boldsymbol{l}\,\boldsymbol{x} = \boldsymbol{l}_m\,\boldsymbol{x} = \boldsymbol{p}\,\boldsymbol{y} \tag{117}$$

Pasinetti (1977, p.130) also asserts that the sum of profits defined in terms of a scalar product of unit-prices of commodities and labour-unit-quantities of wage-commodities may not be equal to the sum of surplus-values defined in terms of a scalar product of unit labour-values of commodities and labour-unit-quantities of wage-commodities. According to Pasinetti, in general,

$$\tau\,\boldsymbol{v}\,\boldsymbol{d}\,\boldsymbol{l}_m\,\boldsymbol{x} \neq \rho\,\boldsymbol{p}\,(\boldsymbol{A}+\boldsymbol{d}\,\boldsymbol{l})\,\boldsymbol{x} \tag{118}$$

This statement may be algebraically correct, but it says nothing about the relation between the sum of profits and the sum of surplus values defined in terms of monetary values or socially necessary labour time.

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Appendix A

We will illustrate the process of reverse transformation of profits into surplus values described by equation (87) using the numerical values from example (38). Since the linear model of production is defined in terms of quantities and the original model of self-reproducing economy presented by Tugan-Baranovsky (1906) is defined in terms of monetary values of production, a vector of unit prices "**p**" needs to be assumed in order to calculate the quantities of commodities.

Table 4: The model of a self-reproducing economy

	Input intermediate goods	**Wages**	**Profits**	**Output**	**Assumed unit price**
Capital goods sector	$180,000,000.00	$60,000,000.00	$60,000,000.00	$300,000,000.00	$10.00
Wage goods sector	$80,000,000.00	$80,000,000.00	$40,000,000.00	$200,000,000.00	$2.00
Luxury goods sector	$40,000,000.00	$60,000,000.00	$25,000,000.00	$125,000,000.00	$5.00

The reproduction schema presented in quantity units and labour time (assuming that the capital goods sector employs 150000 workers and production period is one year.

Table 5: The model of a self-reproducing economy expressed in labour time and quantity units

	Input intermediate goods [units]	**Labour time [worker-years]**	**Output [units]**
Capital goods sector	18,000,000	150,000	30,000,000
Wage goods sector	8,000,000	200,000	100,000,000
Luxury goods sector	4,000,000	150,000	25,000,000

Using matrix notation:

$$\boldsymbol{p} = [10, 2, 5] \tag{119}$$

$$\boldsymbol{x} = \begin{bmatrix} 30000000 \\ 100000000 \\ 25000000 \end{bmatrix} \tag{120}$$

The vector of quantities of net product **y** consists of the quantities of wage and luxury goods only.

$$\boldsymbol{y} = \begin{bmatrix} 0 \\ 100000000 \\ 25000000 \end{bmatrix} \tag{121}$$

The matrix **A** and vector of labour intensities **l** describe the technology in the model. Only the first commodity is required in the production of all the commodities so the second and third rows of matrix **A** are filled with zeros. The coefficients in in the first row correspond to ratios of input to output commodities (measured in commodity units).

$$\boldsymbol{A} = \begin{bmatrix} 0.60, 0.08, 0.16 \\ 0.00, 0.00, 0.00 \\ 0.00, 0.00, 0.00 \end{bmatrix} \tag{122}$$

The vector of labour intensities is populated with direct labour coefficients, corresponding to the ratios of the amounts of direct labour required in the production process (measured in units of labour-time) to the output produced by the sectors (measured in commodity units).

$$\boldsymbol{l} = [200.00, 500.00, 166.67] \tag{123}$$

The profit rate, ρ=0.25 has been determined in equation (39).

We can plug these values into equation (87) to validate the results. To avoid making mistakes, a GNU Octave script has been created. We can see that the value of τ obtained from the linear model of production is the same as that calculated using formula (40).

```
1;
y = [0; 1e+8; 2.5e+7];
```

```
rho = 0.25;
A = [0.6, 0.08, 0.16;0,0,0;0,0,0];
l = [150.0/30000.0,200.0/100000.0,150.0/25000.0];
I=eye(3);
tau=(1+rho)*(l*inv(I-(1+rho)*A)*y)/(l*inv(I-A)*y) – 1

tau =  0.62500
```

Appendix B

We will re-evaluate the numerical example provided by Pasinetti (1977, pp.144-149), which is based on the numerical model of the "wheat-iron-pigs" (or "wheat-iron-turkeys") economy initially introduced by Sraffa (1960, p.4). The technology is defined by specifying the matrix of coefficients of production "**A**" and the vector of labour coefficients "**l**". The MELT "μ" is assumed to be one, as the labour values expressed in labour time and monetary units are equal. The vector of net product "**y**" is given, we also know that workers buy the vector of quantities of commodities "$\mathbf{y}_w$". The following values of the parameters have been chosen:

$$\boldsymbol{A}=\begin{bmatrix}0.413333, 2.571429, 0.500000\\ 0.026667, 0.285714, 0.050000\\ 0.020000, 0.285714, 0.250000\end{bmatrix} \tag{124}$$

$$\boldsymbol{l}=[200.00, 500.00, 166.67] \tag{125}$$

$$\boldsymbol{y}=\begin{bmatrix}180\\ 0\\ 30\end{bmatrix} \tag{126}$$

$$\boldsymbol{y}_w=\begin{bmatrix}180\\ 0\\ 30\end{bmatrix} \tag{127}$$

The sum of prices of commodities consumed by workers is equal to the wage bill "W":

$$W=\boldsymbol{p}\,\boldsymbol{y}_w \tag{128}$$

If the rate of surplus value "τ" is known, the wage rate "w" can be calculated from the following formula:

$$w=\frac{\mu}{1+\tau} \tag{129}$$

We know from equations (77) and (81) that:

$$w=\frac{W}{\Lambda_{lt}}=\frac{Y}{(1+\tau)\Lambda_{lt}}=\frac{\mu\Lambda_{lt}}{(1+\tau)\Lambda_{lt}} \tag{130}$$

We will initially concentrate on calculating the rate of profit "ρ". It is assumed that the input values used in the example are economically meaningful.

The wage bill “W” used in equation (128) is defined in equation (77). We can also substitute the value of “**p**” from equation (75) into equation (128).

$$(1+\rho)w\boldsymbol{l}[\boldsymbol{I}-(1+\rho)\boldsymbol{A}]^{-1}\boldsymbol{y}_w=w\boldsymbol{l}\boldsymbol{x} \tag{131}$$

In equation (131) the wage rate “w” can be eliminated. The vector of quantities of all the commodities produced in the economy “**x**” can be calculated from equation (65) by inverting the matrix (**I-A**).

$$\boldsymbol{x}=(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{y} \tag{132}$$

The value of “**x**” from equation (132) is then substituted into equation (131).

$$\boldsymbol{l}(1+\rho)[\boldsymbol{I}-(1+\rho)\boldsymbol{A}]^{-1}\boldsymbol{y}_w=\boldsymbol{l}(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{y} \tag{133}$$

After rearranging we get the following homogeneous linear system of equations “**lM**(ρ)=**0**”. The unknown variable is the rate of profit “ρ”.

$$\boldsymbol{l}\{(1+\rho)[\boldsymbol{I}-(1+\rho)\boldsymbol{A}]^{-1}\boldsymbol{y}_w-(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{y}\}=\boldsymbol{0} \tag{134}$$

We will find the value of “ρ” satisfying equation (134) by introducing an error function “e(ρ)” equal to the Euclidean norm of the left-hand side of equation (134) and then numerically minimising this function.

$$e(\rho)=\|\boldsymbol{l}\{(1+\rho)[\boldsymbol{I}-(1+\rho)\boldsymbol{A}]^{-1}\boldsymbol{y}_w-(\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{y}\}\|=0 \tag{135}$$

A GNU Octave script has been created to evaluate and validate the results.

```
1;
#input data, the technology is described by A and l
global A = [186/450, 54/21, 30/60;12/450,6/21,3/60;9/450,6/21,15/60]
global l = [18/450,12/21,30/60]
global mu=1                    # MELT
global I=eye(3)
global y=[180;0;30]       # real net product
global y_w=[120;0;10]     # workers' consumption
# end of input data

x=inverse(I-A)*y          # (130)
v=mu*l*inverse(I-A)       # (95),(96)

function ret=e(rho)       # (133)
  global l;
  global I;
  global A;
  global y_w;
  global y;
  ret=norm(l*((1+rho)*inverse(I-(1+rho)*A)*y_w-inverse(I-A)*y));
endfunction

[rho, fval, info] = fsolve (@e, 0)

tau=(1+rho)*(l*inverse(I-(1+rho)*A)*y)/(l*inverse(I-A)*y) - 1 # (87)
w = mu/(1+tau)            # (129)
p=(1+rho)*w*l*inverse(I-(1+rho)*A) # (75)
p_rel = p/p(1)            # relative unit prices expressed as fractions of p(1)
```

```
Y=p*y                    # (76) the sum of prices of production
LAMBDA=v*y               # (91)

S= tau/(1+tau) * v * y   # (88)
P= rho * (w*l*x + p*A*x) # (89)
```

We can compare the results obtained from the script with the data presented by Pasinetti (1977)

Table 6: Comparison of the results from the Marxian and Neo-Ricardian models

	Results from the GNU Octave script	**Data provided by Pasinetti (1977)**
Labour values "**v**"	[0.18182, 1.81818, 0.90909]	[0.1818, 1.81818, 0.90909]
Rate of surplus value "τ"	0.86431	0.9411
Rate of profit "ρ"	0.18537	0.1854
Relative unit prices expressed as a fraction of p_1	[1.0000, 9.2861, 3.8495]	[1, 9.286, 3.849]
Unit prices "**p**"	[0.20306, 1.88561, 0.78166]	[0.1950, 1.8109, 0.7507]
Wage rate "w"	0.53639	0.515
Sum of prices of production of net social product "Y"	60	
Sum of labour values of net social product "Λ"	60.000	
Sum of profits "P"	27.817	
Sum of surplus values "S"	27.817	

We can see that the sum of surplus values "S" is equal to the sum of profits "P" and that the sum of the prices of production of the net social product "Y" is equal to the sum of its values "Λ"

The vector of "Marxian" labour values "**v**" and the rate of profit "ρ" are the same as in the original solution provided by Pasinetti (1977). The ratios of prices of individual commodities "p_1:p_2:p_3" are also (almost) identical but since Pasinetti chooses a different "numeraire", the wage rate "w" and the actual unit prices "**p**" are different. We have demonstrated that the Neo-Ricardian Labour Theory of Value significantly differs from the Marxian Labour Theory of Value and that it is possible to find a solution that is entirely consistent with the Marxian theory to the model of a self-reproducing system, defined by Sraffa (1960) and Pasinetti (1977).